Year 12 Mathematics Extension 1

Vectors

Vectors and Their Properties

MATHEMATICS

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First Name .... Last Name ....

MEV1

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This Lesson covers the following points from NESA Syllabus:

Introduction to vectors

1. Define a vector as a quantity having both magnitude and direction, and examine examples of vectors, including displacement and velocity:
   - explain the distinction between a position vector and a displacement (relative) vector

2. Define and use a variety of notations and representations for vectors in two dimensions:
   - use standard notations for vectors, for example: $\underset{\sim}{a}$, $\overrightarrow{AB}$ and $\mathbf{a}$
   - represent vectors graphically in two dimensions as directed line segments
   - define unit vectors as vectors of magnitude 1, and the standard two-dimensional perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$
   - express and use vectors in two dimensions in a variety of forms, including component form, ordered pairs and column vector notation

3. Perform addition and subtraction of vectors and multiplication of a vector by a scalar algebraically and geometrically, and interpret these operations in geometric terms**:**
   - graphically represent a scalar multiple of a vector
   - use the triangle law and the parallelogram law to find the sum and difference of two vectors
   - define and use addition and subtraction of vectors in component form
   - define and use multiplication by a scalar of a vector in component form

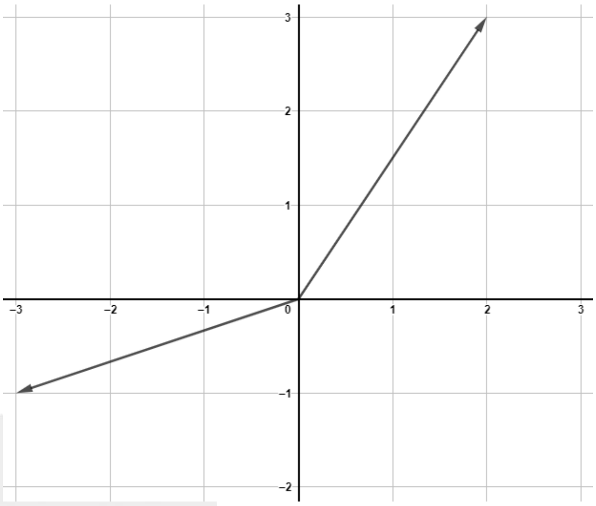
V1.1 –What is a Vector?

A vector is a geometrical object with two properties. It has a ‘distance’ and a ‘direction’.
They are usually in written in the forms:

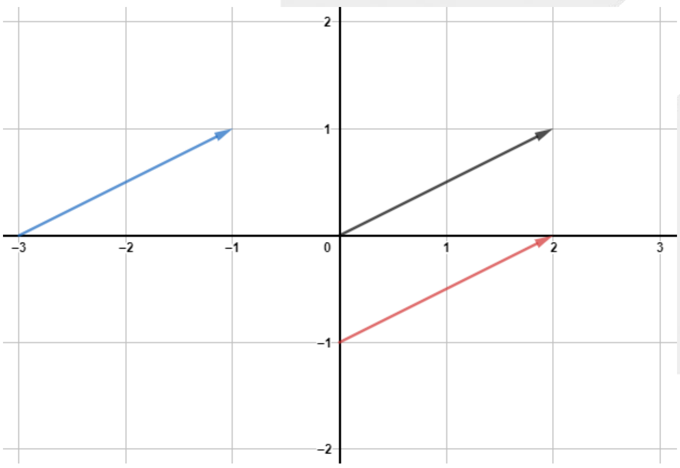
$$\underline{v} = \vec{v} = \boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Naturally, we call x the x-component and y the y-component.

We may also sometimes write vectors like $\vec{v} = (x, y)$ or as $\vec{v} = [x \quad y]$
This gives rise to the geometric interpretation of vectors. Every vector can be assigned a unique point on the ‘Cartesian Plane’, with its tail at $(0,0)$ and its head at (x, y).
Here are the vectors $(2,3)$ and $(-3,-1)$ in their standard form.



However, vectors are only defined by their distance and direction, not by the position of their ‘root’. We can move vectors around the plane without changing the vector itself.
Consider the following example:



The vector on the left goes from:
$(-3,0)$ to $(-1,1)$.
By shifting the vector across three,
This is equivalent to $(2,1)$.

In essence, moving vectors on the place preserves the vectors, as long as the direction and distance is the same.
Vectors whose tail is off the origin are called ‘displacement or relative vectors’.

Given two points $A(x_a, y_a)$ and $B(x_b, y_b)$, we can define the vector $\overrightarrow{AB}$ which goes from the point A to the point B. By shifting this vector to the origin, we can see that:

$$\overrightarrow{AB} = (x_b - x_a, y_b - x_a)$$

Note, in the Mathematics Extension 1 course, we will focus on examining vectors in two dimensions, (also known as $\mathbb{R}^2$), but some of the extension questions will concern 3D vectors (or even n-dimensional) vectors.

Question 1

Find the following vectors in the form (x, y) and draw the given vectors on the Cartesian Plane.

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

c) The vector from the origin to the point $(2, -3)$.

d) The vector from the point $(3,2)$ to the point $(1,1)$.

e) The vector from the point $(1,2)$ to the origin.

f) The vector with length 5 and travelling in the positive horizontal direction from $(-3,1)$.

V1.2 – Addition of Vectors

Vector addition is defined in the expected way:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

This is known as 'component-wise' addition, because we add each component separately.
This gives rise to many algebraic and geometric properties which we will investigate through the lesson and in the worksheets.

Firstly, the 'zero-vector' (written as $\vec{0}$ or $\mathbf{0}$), is the vector such that for any vector $\vec{v}$:

$$\vec{v} + \vec{0} = \vec{v}$$

This implies that in 2D, $\vec{0} = (0,0)$, such that:

$$(x, y) + (0,0) = (x + 0, y + 0) = (x, y)$$

Furthermore, every vector has an 'opposite', such that $\boldsymbol{v} + \boldsymbol{v}' = \mathbf{0}$.
For some vector (x, y), let (x', y') be the opposite vector. Hence, we expect that:

$$(x, y) + (x', y') = (0,0)$$

$$(x + x', y + y') = (0,0)$$

And by solving for each component:

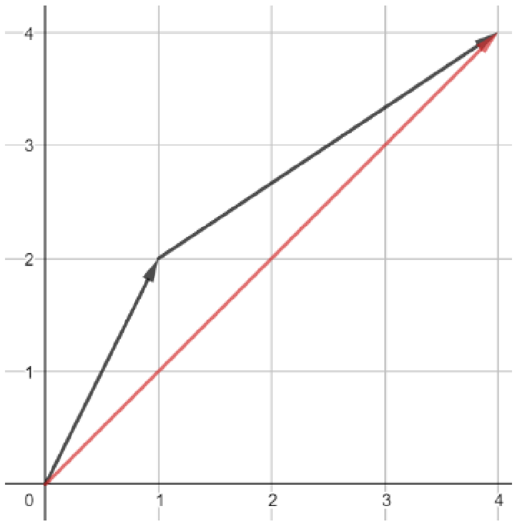
$$x' = -x \text{ and } y' = -y$$

So, the opposite of any vector (x, y) is the vector $(-x, -y)$.

Geometrically, this vector will have the same direction but opposite direction. We write this as $-\boldsymbol{v}$.

Finally, the addition of vectors on the plane is done by the 'tip-to-tail' method.

Consider $(x, y) = (1,2) + (3,2)$.



We shift the tail of the vector $(3,2)$ to the tip of the vector we are adding to, namely $(1,2)$.

Hence the resultant vector is the 'overall' vector from the base of the first vector to the tip of the second vector.

We can see that this is $(x, y) = (1 + 3, 2 + 2) = (4,4)$ which is the vector we see in the diagram.

This also gives us another way to find the vector $\overrightarrow{AB}$. Recall that we found that,

$$\overrightarrow{AB} = (b_x - a_x, b_y - a_y)$$

$$= (b_x, b_y) - (a_x, a_y)$$

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

Where $\overrightarrow{OP}$ is the vector from the origin to the point $P(p_x, p_y)$, which by definition is simply the vector (p_x, p_y). We can define vector subtraction in this way with $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.

Question 2

Let $a = (1,3)$, $b = (2,1)$, $c = (-1,-3)$, $d = (-2,0)$.

a) Graph all above vectors on the Cartesian Plane.

b) Find the vector $a + c$.

c) Find the vector $d - b$.

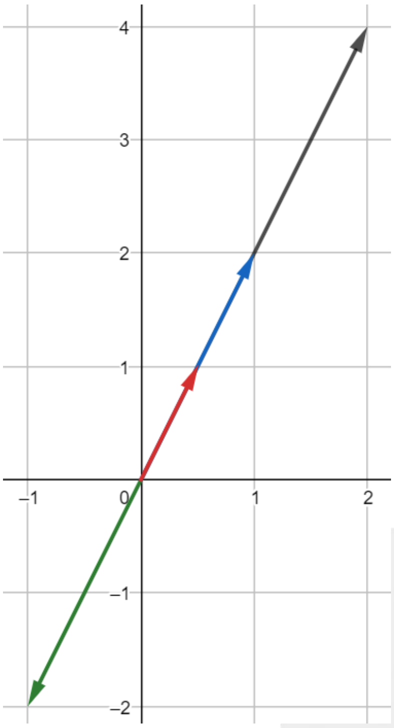
d) Find the vector $(a + c) - (d - b)$. Prove this is equal to $a + b + c - d$.

e) Find the vector $\overrightarrow{BA}$, where A is the point at the tip of the vector a based at the origin.

f) Similarly, if $e = a + b$, find the vector $\overrightarrow{DE}$ and plot it with its tail at the origin.

V1.3 – Scalar Multiplication

Scalar multiplication takes a real number and a vector, and ‘stretches’ the length of the vector by that number but keeps it’s direction. Component-wise, we define it as:



$$k\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}kx\\ky\end{bmatrix}$$

For example, consider the vector $\boldsymbol{v}=(1,2)$.
We show the vectors $2\boldsymbol{v}, 0.5\boldsymbol{v}$ and $(-1)\boldsymbol{v}$.
Label the vectors.

We can also make the following observations:
Let $\boldsymbol{v}$ be a vector.

- $0\boldsymbol{v}$ is the zero vector $\boldsymbol{0}$
- The negative vector $-\boldsymbol{v}$ is the same as $(-1)\boldsymbol{v}$
- Given two scalars: λ, μ, then $(\lambda+\mu)\boldsymbol{v}=\lambda\boldsymbol{v}+\mu\boldsymbol{v}$
- Given two vectors: $\boldsymbol{v}, \boldsymbol{w}$, then $\lambda(\boldsymbol{v}+\boldsymbol{w})=\lambda\boldsymbol{v}+\lambda\boldsymbol{w}$

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It will be optional to prove these properties in the homework. In fact, they work for any space of vectors (see: 1st Year University).

We can now define any vector in terms of the ‘standard basis’. We have the vectors:

$$\boldsymbol{i}=\begin{bmatrix}1\\0\end{bmatrix} \text{ and } \boldsymbol{j}=\begin{bmatrix}0\\1\end{bmatrix}$$

Then for any vector $\boldsymbol{v}=(a,b)$, then we have $\boldsymbol{v}=a\boldsymbol{i}+b\boldsymbol{j}$. Check this below with our definitions:

In fact, consider two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ where $\boldsymbol{w}\neq k\boldsymbol{v}$ (e.g. they are not parallel or zero), then any other vector $\boldsymbol{x}$ can be written as $\boldsymbol{x}=a\boldsymbol{v}+b\boldsymbol{w}$ for some unique numbers a, b.

We call these two vectors a ‘basis’ of the 2D real vectors, and $\boldsymbol{i}, \boldsymbol{j}$ the ‘standard basis’.
This can also be extended for n-dimensions, where:

$$\begin{bmatrix}k_1\\k_2\\\vdots\\k_n\end{bmatrix}=k_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}+\cdots+k_n\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

We can also define ‘unit vectors’, which have a length of 1. We write $\|\boldsymbol{v}\|$ to denote the length of a vector (think of absolute value). From any non-zero vector, the unit vector in that direction is:

$$\hat{\boldsymbol{v}}=\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Of course, we haven’t learnt to calculate vector length yet, but we will do that next week.

Question 3

Let $\boldsymbol{a} = (-1, -2)$, $\boldsymbol{b} = (0,4)$ and $\boldsymbol{c} = (1, -1)$.

a) Compute $2\boldsymbol{a}$ and $-3\boldsymbol{c}$.

b) Compute $3\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$.

c) Plot the vectors $-2\boldsymbol{a}, 0.5\boldsymbol{b}$ and $0.5\boldsymbol{b} - 2\boldsymbol{a}$ on a plot.

d) Plot the standard basis vectors, and graphically show how $v = (2,3) = 2\boldsymbol{i} + 3\boldsymbol{j}$.

e) By solving simultaneous equations, write $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ in the form $a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Practice Questions and Further Important Ideas

Question 4

a) Plot the vectors $(2,-1)$, $(3,1)$ and $(-1,0)$ on the Cartesian Plane.

b) Plot two arbitrary non-zero, non-parallel vectors $\boldsymbol{v}, \boldsymbol{w}$.
Then plot $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{w}+\boldsymbol{v}$ to show they are equal.
(This is known as the parallelogram law).

c) Similarly, plot $\boldsymbol{v}$ and $\boldsymbol{w}$ and then plot $\boldsymbol{w}-\boldsymbol{v}$ with its tail at $\boldsymbol{v}$.
Use this diagram to show that $\boldsymbol{w}-\boldsymbol{v}=\boldsymbol{w}+(-1)\boldsymbol{v}$.
(This is known as the triangle law).

d) Graph the vector between from the point $(8,2)$ to $(3,-1)$ and show this vector in standard position.

Question 5

a) Write the vector from the origin to the point $(3,4)$ in all the forms you can remember.

b) If $\boldsymbol{a} = (3, -2)$ and $\boldsymbol{b} = (2, k)$, find the value of k so that $2\boldsymbol{a} - 3\boldsymbol{b}$ is the zero vector.

c) If $\boldsymbol{v} = 2\begin{bmatrix} 1 \\ -3 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, then write the vector $\boldsymbol{v}$ in standard basis form.

d) If $\boldsymbol{v}$ is the vector from $(4,2)$ to $(9,1)$ and $\boldsymbol{w}$ is the vector from $(-2,0)$ to $(-5,-2)$, then find the vectors $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$.

e) By using a diagram, find the distance between the origin and the point $(3,2)$. Hence, find the length of this vector and so find the unit vector in this direction.

f) Write the standard basis for the vectors of real numbers in three dimensions.

myPrime

6 Spring St Chatswood | www.primeeducation.com.au | (02) 9410 1074